Towards a geometrical classification of statistical conservation laws in turbulent advection.

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Abstract

The paper revisits the compressible Kraichnan model of turbulent advection in order to derive explicit quantitative relations between scaling exponents and Lagrangian particle configuration geometry.

1. Introduction

Physical and numerical experiments have typically access to, at least, two kinds of observables of the state of a turbulent Newtonian fluid. These observables are, on the one hand, Eulerian or Lagrangian statistical indicators such as universal scaling properties of correlation and structure functions. On the other hand, geometrical indicators like shapes formed by configurations privileged by the dynamics of Lagrangian or inertial particles can be observed. A series of remarkable results (see e.g. [1, 2] and references therein) have in recent years started to unveil the nature of the link between statistical indicators and geometrical properties. In particular, [3] gave clear evidence that tracer particles advected by the $2d$ Navier–Stokes velocity field spend anomalously long times in degenerate geometries characterized by strong clustering and that the phenomenon has a quantitative counterpart in the existence of special functions of particle configurations which are on average preserved by flow. Furthermore, it was shown in [4] for densities transported by compressible turbulent velocity fields that statistical conservation laws determine, at least for integer values of the mass, the multifractal spectrum of the attractor towards which Lagrangian trajectories converge. On the background of these results was the discovery of anomalous scaling [5, 6, 7] in a stylized model of passive advection, the Kraichnan model [8]. There it was possible to show in a mathematically controlled fashion [9], that the universal statistical properties of the model in the inertial and, under additional hypotheses [10], decay ranges are determined by statistical conservation laws of the Lagrangian dynamics (referred to as zero modes). Zero modes have subsequently become the object of extensive investigations and analytical, mostly perturbative, and numerical expressions of their scaling exponents have been derived for the Kraichnan model and its generalizations (see e.g. [11, 12] and references therein). To the best of my knowledge, however, quantitative expressions of the scaling exponents explicitly relating them to geometrical properties of Lagrangian particle configurations appeared only recently in [13]. The scope of the present contribution is to illustrate the new method of calculation introduced in [13] in the slightly more general case of the compressible Kraichnan model [14]. The main result of the paper, derived in sections 4 and 5, is the classification of zero modes, irreducible and reducible, in terms of Gel'fand-Zetlin patterns (see e.g. [15]) associated to quadratic Casimir operators of classical groups. It must be clearly stated that this classification

is perturbative but it has nevertheless the merit to establish an explicit link to Lagrangian particle geometry which is discussed in more details for four point correlations in section 6. Sections 2 and 3 recall respectively the defining properties of the Kraichnan model and basic facts about the structure of the solution of the Hopf equations. In the conclusions I discuss the use of the new method and make some speculations on its significance for the Navier–Stokes equation as well as for other statistical field theories.

2. The compressible Kraichnan model

Passive turbulent advection by a *compressible* velocity field in $\mathbb{R}^d$ encompasses (see e.g. [14]) the density evolution equation

$$\partial_t \rho + \partial_{\boldsymbol{x}} \cdot (\boldsymbol{v}\rho) = \frac{\kappa}{2}\partial_{\boldsymbol{x}}^2 \rho + f \tag{1}$$

(describing e.g. the distribution of particles floating on the surface of the fluid in the limit of vanishing Stokes number) and the tracer evolution equation

$$\partial_t \theta + \boldsymbol{v} \cdot \partial_{\boldsymbol{x}} \theta = \frac{\kappa}{2}\partial^2 \theta + g \tag{2}$$

(describing e.g. the fluid temperature). At finite molecular viscosity κ, the Lagrangian dynamics underlying (1) and (2) are distinct, respectively forward and backward in time. In the *inviscid limit* and in the case of advection by a turbulent field stylized by a stationary, time-decorrelated Gaussian ensemble they become formally adjoint [14] and can be discussed in parallel. These are the working hypotheses of the paper. In particular, the velocity field in (1), (2) is modeled by the realizations of Kraichnan's zero average Gaussian compressible ensemble:

$$\prec v^{\alpha}(\boldsymbol{x}_1, t_1) v^{\beta}(\boldsymbol{x}_2, t_2) \succ = \delta(t_{12}) D^{\alpha\,\beta}(\boldsymbol{x}_{12}, m) \tag{3a}$$

$$D^{\alpha\,\beta}(\boldsymbol{x}; m) = D_0\, \xi\, (2-\xi) \int \frac{d^d p}{(2\,\pi)^d} \frac{e^{\imath \boldsymbol{p}\cdot\boldsymbol{x}}}{p^{d+\xi}} \left\{ (1-\wp)\delta^{\alpha\,\beta} - (1-d\,\wp)\frac{p^{\alpha}p^{\beta}}{p^2} \right\} \chi\left(\frac{m^2}{p^2}\right) \tag{3b}$$

where $\boldsymbol{x}_{12} := \boldsymbol{x}_1 - \boldsymbol{x}_2$, $t_{12} := t_1 - t_2$, $\xi \in [0,2]$ is the roughness degree of the velocity field, $\wp \in [0,1]$ is the degree of compressibility and χ some non-universal infra-red cut-off function normalized to the unity for vanishing inverse integral scale m. The force fields f, g (Gaussian, zero-average and decorrelated in time) compensate the dissipation, in the inviscid limit only due to the eddy diffusivity generated by the velocity field, in order to let the system attain a Galilean invariant steady state. Consistence with hydrodynamics imposes to interpret (1), (2) advected by the Kraichnan ensemble as stochastic partial differential equations in the sense of Stratonovich. The role of the eddy diffusivity appears upon converting (1), (2) into Ito form [11, 12]. This latter is for the density

$$\partial_t \rho + \partial_{\boldsymbol{x}} \cdot (\boldsymbol{v}\rho) = \frac{\varkappa\, m^{-\xi}}{2}\partial^2 \rho + f \tag{4}$$

and for the tracer

$$\partial_t \theta + \boldsymbol{v} \cdot \partial_{\boldsymbol{x}} \theta = \frac{\varkappa\, m^{-\xi}}{2}\partial^2 \theta + g \tag{5}$$

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The eddy diffusivity in the above equations is specified by

$$\varkappa\, m^{-\xi} = \frac{D^{\alpha}_{\ \alpha}(\mathbf{0}, m)}{d} = \frac{d-1}{d} \int \frac{d^d p}{(2\pi)^d} \frac{D_0\, \xi}{p^{d+\xi}} \chi\left(\frac{m^2}{p^2}\right) \tag{6}$$

which means that Taylor's formula [16] becomes exact in the Kraichnan model.

3. Hopf equations and martingales

A straightforward application of Ito calculus to (2) shows that the equal time correlation $\mathcal{C}_n$ of n tracer fields in $\mathbb{R}^d$ satisfies the Hopf equation (see e.g. [9, 11])

$$\left(\partial_t + \mathcal{M}^{(n)}_{\boldsymbol{X}}\right) \mathcal{C}_n\left(\boldsymbol{X}; t\right) = \mathcal{F}_n\left(\boldsymbol{X}; t\right) \tag{7}$$

with

$$\mathcal{M}^{(n)}_{\boldsymbol{X}} = -\frac{1}{2} \sum_{ij}^{n} D^{\alpha\,\beta}(\boldsymbol{x}_{ij}, m) \partial_{x^{\alpha}_{i}} \partial_{x^{\beta}_{j}} \tag{8}$$

and $\mathcal{F}_n$ an effective forcing term fully specified by the correlation of g and by correlation functions $\mathcal{C}_{n'}$ with $n' < n$. The operator $\mathcal{M}_n$ can be also regarded as the generator of a multiplicative diffusion process the transition probability $\mathcal{P}_n$ whereof obeys

$$\left(\partial_t + \mathcal{M}^{(n)}_{\boldsymbol{X}}\right) \mathcal{P}_n\left(\boldsymbol{X}, \boldsymbol{X}'; t - t'\right) = 0 \tag{9a}$$

$$\lim_{t \downarrow t'} \mathcal{P}_n\left(\boldsymbol{X}, \boldsymbol{X}'; t - t'\right) = \delta^{(n\,d)}(\boldsymbol{X} - \boldsymbol{X}') \tag{9b}$$

The Lagrangian interpretation of $\mathcal{P}_n$ is that it is the probability density to find a cluster of n inertial particles in $\boldsymbol{X}' = (\boldsymbol{x}'_1, \dots, \boldsymbol{x}'_n)$ at time t' conditioned upon the event that they reach $\boldsymbol{X} = (\boldsymbol{x}_1, \dots, \boldsymbol{x}_n)$ at a later time t. Analogously, the equal time n-point correlation of the density field satisfies an Hopf equation formally adjoint to (7). In such a case $\mathcal{P}_n$ can be regarded as the solution of the Fokker-Planck equation

$$\left(\partial_t + \mathcal{M}^{(n)\dagger}_{\boldsymbol{X}'}\right) \mathcal{P}_n\left(\boldsymbol{X}, \boldsymbol{X}'; t - t'\right) = 0 \tag{10}$$

with

$$\mathcal{M}^{(n)\dagger}_{\boldsymbol{X}'} = -\frac{1}{2} \sum_{ij}^{n} \partial_{x^{\alpha}_{i}} \partial_{x^{\beta}_{j}} D^{\alpha\,\beta}(\boldsymbol{x}_{ij}, m) \tag{11}$$

$\mathcal{P}_n$ acquires the interpretation of the probability density for a cluster of n inertial particle to *arrive* in $\boldsymbol{X}' = (\boldsymbol{x}'_1, \dots, \boldsymbol{x}'_n)$ at time t conditioned upon the event that they are in $\boldsymbol{X} = (\boldsymbol{x}_1, \dots, \boldsymbol{x}_n)$ at time $t' \leq t$. The different interpretations of (9a) and (10) stem from the distinct Lagrangian dynamics underlying (1), (2) at finite molecular viscosity [14]. Whenever the kernel $\mathcal{P}_n$ acts on translational invariant functions, the arithmetic average of the Lagrangian particle positions is integrated out. The reduction defines a $d_n = (n-1)\, d$-dimensional subspace of $\mathbb{R}^{n\,d}$, the translational invariant sector of the theory. The projection P_n of $\mathcal{P}_n$ on the subspace has a well defined

scale invariant limit for m tending to zero [9]. The scaling dimensions of P_n corresponding to a linear rescaling of spatial variables (i.e. in units d_x) are

$$\mathsf{d}_t = (2-\xi)\,\mathsf{d}_x \qquad \& \qquad \mathsf{d}_{P_n} = -(n-1)\,d\,\mathsf{d}_x \equiv -d_n\,\mathsf{d}_x \tag{12}$$

Qualitative analysis [9] yields for P_n the asymptotic expansion:

$$P_n(\boldsymbol{Y},\boldsymbol{Y}',t-t') = \begin{cases} \sum_{i=0}^{\infty} \phi_{i;1}\,(\boldsymbol{Y})\,\psi_{i;1}\left(\boldsymbol{Y}',t-t'\right) & \text{for } Y \le Y' \\ \sum_{i=0}^{\infty} \psi_{i;2}\,(\boldsymbol{Y},t-t')\,\phi_{i;2}\left(\boldsymbol{Y}'\right) & \text{for } Y > Y' \end{cases}, \qquad \boldsymbol{Y},\boldsymbol{Y}' \in \mathbb{R}^{d_n} \tag{13}$$

Throughout the manuscript, H_a denotes the Heaviside function with normalization $H_a(0) = a$. The meaning of the expansion is well illustrated by the $\xi = 0$ limit. In such a case

$$\mathcal{M}^{(n)}_{\boldsymbol{X}} = \mathcal{M}^{(n)\dagger}_{\boldsymbol{X}} = -\frac{\varkappa}{2}\partial^2_{\boldsymbol{X}} \equiv -\frac{\varkappa}{2}\Delta_{n\,\boldsymbol{X}}$$

and both $\mathcal{P}_n$ and the reduced propagator P_n are Gaussian. Self-adjointness of the Laplacian also implies

$$\psi_i = \psi_{i;1} = \psi_{i;2} \qquad \& \qquad \phi_i = \phi_{i;1} = \phi_{i;2} \tag{14}$$

It is expedient to couch the sum over i into a sum over $(\jmath,\boldsymbol{l},k)$, $\jmath$ being the degree of homogeneity of an harmonic polynomial $\mathcal{H}_{\jmath\boldsymbol{l}}$, $\boldsymbol{l}$ the numbers specifying an $SO(d)$-adapted representation $\mathcal{Y}_{\jmath\boldsymbol{l}}$ of hyperspherical harmonics of $SO(d_n)$ [17] and

$$\phi_{\jmath\boldsymbol{l}k}(\boldsymbol{Y}) := \frac{Y^{2\,k}\,\mathcal{H}_{\jmath\boldsymbol{l}}(\boldsymbol{Y})}{2^{2k}\,\Gamma(k+1)\,\Gamma\left(\frac{d_n+2\,\jmath+2k}{2}\right)} = \frac{Y^{2\,k+\jmath}\,\mathcal{Y}_{\jmath\boldsymbol{l}}(\boldsymbol{Y}/Y)}{2^{2k}\,\Gamma(k+1)\,\Gamma\left(\frac{d_n+2\,\jmath+2k}{2}\right)} \tag{15}$$

for $Y \equiv ||\boldsymbol{Y}||$. The so defined $\phi_{\jmath\boldsymbol{l}k;1}$'s satisfy the "tower" relations

$$\partial^2_{\boldsymbol{Y}}\phi_{\jmath\boldsymbol{l}k}(\boldsymbol{Y}) = \phi_{\jmath\boldsymbol{l}k-1}(\boldsymbol{Y}), \qquad \phi_{\jmath\boldsymbol{l}\,-1}(\boldsymbol{Y}) = 0 \tag{16}$$

Under the same conventions, the identity

$$\psi_{\jmath\boldsymbol{l}k}(\boldsymbol{Y},t) = 2\,\pi^{\frac{d_n}{2}}\,\frac{H_{\jmath\boldsymbol{l}}(\boldsymbol{Y})\,e^{-\frac{R^2}{2\,\varkappa\,t}}\,L_k^{\left(\frac{d_n+2\,\jmath-2}{2}\right)}\left(\frac{R^2}{2\,\varkappa\,t}\right)\Gamma\left(k+1\right)}{2^{\jmath-k}\,(2\,\pi\,\varkappa\,t)^{\frac{d_n}{2}}\,(\varkappa\,t)^{\jmath+k}} \tag{17}$$

holds true for $L_b^{(a)}$ the generalized Laguerre polynomial of degree k. An alternative useful representation is

$$\psi_{\jmath\boldsymbol{l}k}(\boldsymbol{Y},t) = 2\,\pi^{\frac{d_n}{2}}\,\frac{\langle\boldsymbol{Y},\varkappa\,t|\,|\jmath\,\boldsymbol{l}\,k\rangle\;e^{-\frac{Y^2}{4\,\varkappa\,t}}}{2^{\jmath-k-1}\,(2\,\pi\,\varkappa\,t)^{\frac{d_n}{2}}\,(\varkappa\,t)^{\jmath+k}} \tag{18}$$

with $\langle\boldsymbol{Y},\varkappa\,t|\,|\jmath\,\boldsymbol{l}\,k\rangle$ the wave function of the $(\jmath,\boldsymbol{l},k)$ eigenstate of the isotropic quantum harmonic oscillator of unit mass and frequency $\varkappa\,t$ in $\mathbb{R}^{d_n}$[18]. Finally, the $\psi_{\jmath\boldsymbol{l}k}(\boldsymbol{Y},t)$'s satisfy the "tower" relations

$$\partial_t\psi_{\jmath\boldsymbol{l}k}(\boldsymbol{Y},t) = \frac{\varkappa}{2}\partial^2_{\boldsymbol{Y}}\psi_{\jmath\boldsymbol{l}k}(\boldsymbol{Y},t) = -\psi_{\jmath\boldsymbol{l}k+1}(\boldsymbol{Y},t) \tag{19}$$

Some remarks are in order.

i The identities (16), (19) are particular case of consistence conditions of the expansion (13) holding for generic ξ [9, 11].

ii By (18) and the orthonormality of quantum eigenstates it readily follows that the harmonic polynomial $\phi_{\jmath \boldsymbol{l} 0}$ is $\mathbb{L}^2(\mathbb{R}^{d_n})$-orthogonal to the $\psi_{\jmath \boldsymbol{l} k}$'s for any $k > 1$. The normalization of the (15)'s is chosen such that they are exactly replicated whenever averaged over the transition probability P_n. In other words, they are martingales of the Wiener process [19].

iii The time integral

$$\int_0^\infty dt\, \psi^{(0)}_{\jmath \boldsymbol{l} k}(\boldsymbol{Y}, t) \propto \frac{H_{\jmath \boldsymbol{l}}(\boldsymbol{Y})}{Y^{d_n + 2\,(\jmath + k - 1)}} \tag{20}$$

defines for k equal zero only *local* martingales (i.e. harmonic functions). The singularity for $Y \downarrow 0$, obstructs (20) absolute integrability and therefore the martingale property. This fact has important consequences for perturbative investigation of anomalous scaling in the decay range (scales larger than the integral scale of the forcing for infinite integral scale of the velocity field [13]).

4. Scaling, martingales of the diffusion process and perturbation theory

The term martingales [19] denotes functions of a Markov process the shape of which, on average, is preserved by the time evolution:

$$\int d^d x\, \mathcal{F}(\boldsymbol{x}, s)\, K\,(\boldsymbol{x}, s|\boldsymbol{x}', s') = \mathcal{F}(\boldsymbol{x}', s') \tag{21}$$

The kernel K in (21) is a probability density for $\boldsymbol{x}$ at time s conditioned upon the occurrence of $\boldsymbol{x}'$ at time s'. For forward (backward) dynamics $s > s'$ ($s' > s$). Clearly, a time independent $\mathcal{F}(\boldsymbol{x})$ satisfying (21) is on average a conserved quantity of the diffusion process described by the transition probability K. Furthermore, (21) requires only integrability of $\mathcal{F}(\boldsymbol{x})$ with respect to the transition probability. The functional space where statistical conservation laws can be sought is therefore larger than that of the physical solutions of the Hopf equations. Turning the attention to the dual action of K, the role of constant martingales is played by stationary pseudo-measure $\mathcal{G}$ satisfying

$$\int d^d x'\; K\,(\boldsymbol{x}, s|\boldsymbol{x}', s')\, \mathcal{G}(\boldsymbol{x}') = \mathcal{G}(\boldsymbol{x}) \tag{22}$$

The relevance of these considerations for the Kraichnan model [9, 11] is that the expansion (13) generically comprises constant martingales $\mathcal{Z}_{i;1}$and stationary pseudo-measures $\mathcal{Z}_{i;2}$. They satisfy the zero mode equations

$$\mathcal{M}^{(n)} \mathcal{Z}_{i;1} = 0 \qquad \& \qquad \mathcal{M}^{(n)\dagger} \mathcal{Z}_{i;2} = 0 \tag{23}$$

and asymptotically dominate in the inertial range the scaling of respectively the tracer and density correlation functions. Observation *iii* at the end of the previous section evinces that the concept of statistical conservation law (i.e. martingale or stationary pseudo-measure) is stronger than the one of zero mode as the (23)'s provide only necessary conditions. For small ξ, small scale zero modes are deformations of harmonic polynomials which are statistical conservation laws of

the Wiener process. In such a case the identification of zero modes with statistical conservation laws holds true. The fact suggests to use (21), (22) to device a perturbative scheme to determine the scaling dimensions $\zeta_{i;r}$ of zero modes $\mathcal{Z}_{i;r}$. Instead of solving perturbatively (23) as in [5, 20], it is possible to read the dependence of scaling dimensions upon ξ by looking at how the transition probability density deforms statistical conservation laws of the Gaussian theory. If statistical conservation laws exist for small but finite values of the roughness parameter, the deformation introduces a logarithmic time dependent prefactor in front of harmonic polynomials. Conservation laws at finite ξ must cancel this time dependence by containing order by order in ξ suitable logarithmic counter-terms in the spatial variables. The constant prefactor of these terms yields the coefficients of the Taylor expansion of the $\zeta_{i;r}$'s. In formulas, the claim is that choosing the state numbers $\boldsymbol{J} = (j, \boldsymbol{l})$ so that the action of the transition probability is within logarithmic accuracy diagonal then

$$\tilde{\mathcal{Z}}_{\boldsymbol{J}} = e^{-t\,\mathcal{M}_n} \circ \mathcal{H}_{\boldsymbol{J}} \overset{t\uparrow\infty}{\sim} \left(\frac{X}{(\varkappa\, t)^{\frac{1}{2-\xi}}} \right)^{\zeta_{\boldsymbol{J}}(\xi) - \zeta_{\boldsymbol{J}}(0)} \mathcal{H}_{\boldsymbol{J}} + \text{subleading or amplitude terms} \tag{24}$$

Since the perturbative expansion of a statistical conservation law $\mathcal{Z}_i$ cannot contain any time t dependence, the scaling dimensions of $\mathcal{Z}$ and $\tilde{\mathcal{Z}}$ must satisfy

$$\mathsf{d}_{\mathcal{Z}} = \mathsf{d}_{\tilde{\mathcal{Z}}} + [\zeta_{\boldsymbol{J}}(\xi) - \zeta_{\boldsymbol{J}}(0)] \tag{25}$$

5. Calculation of scaling dimensions

As solution of (9a) at leading order in perturbation theory in ξ the transition probability density $\mathcal{P}_n$ is

$$\mathcal{P}_n = e^{-t\,\mathcal{M}_n} = \\ e^{\frac{\varkappa\, t}{2}\Delta_n} - \xi \frac{\varkappa\, t\, \ln m}{2} \Delta_n e^{\frac{\varkappa\, t}{2}\Delta_n} + \frac{\xi}{2} \sum_{i \neq j} (R_2^{(1)})_{ij} (e^{\frac{\varkappa\, t}{2}\Delta_{n-2}})_{n/(ij)} + O(\xi^2) \tag{26}$$

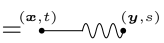
The subscript $n/(ij)$ betokens dependence upon all n particles coordinates but the i-th and the j-th particle.

$$R_2^{(1)}(t) := H_o(t) \left. \frac{d}{d\xi} \right|_{\xi=0} e^{-t\,\mathcal{M}_2} \tag{27}$$

is the first order correction to the two-points response function.

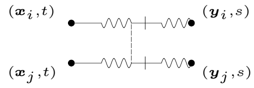
5.1. *Diagrammatic expression of* $R_2^{(1)}$

$R_2^{(1)}$ can be computed by standard diagrammatic techniques of statistical field theory (see e.g. [12]). Denoting for $t > s$

$e^{\frac{\varkappa\,\Delta_1}{2}} =$  & $D^{\alpha\,\beta}_{(1)} := \left. \frac{d}{d\xi} \right|_{\xi=0} D^{\alpha\,\beta} =$

and derivatives by a line perpendicular to response lines, $R_2^{(1)}$ is given by

$$R_2^{(1)}(\boldsymbol{x}_i, \boldsymbol{x}_j, t|\boldsymbol{y}_i, \boldsymbol{y}_j, s) =$$



$$= \int \prod_{r=0}^{2} \frac{d^d p_r}{(2\,\pi)^d}\, e^{\imath \sum_{l=0}^{2} \boldsymbol{p}_l \cdot \boldsymbol{r}_l} e^{-\frac{(t-s)\,\varkappa}{2}\,(\sum_{l=1}^{2} p_l^2)}\, \tilde{R}_2^{(1)}(\boldsymbol{p}_0, \boldsymbol{p}_1, \boldsymbol{p}_2, t-s) \tag{28}$$

with

$$\tilde{R}_2^{(1)}(\boldsymbol{p}_0, \boldsymbol{p}_1, \boldsymbol{p}_2, t) := \frac{p_{1\,\alpha}\, p_{2\,\beta} \left(1 - e^{-t\,\varkappa\,[p_0^2 + \boldsymbol{p}_0 \cdot (\boldsymbol{p}_1 - \boldsymbol{p}_2)]}\right)}{\varkappa\,[p_0^2 + \boldsymbol{p}_0 \cdot (\boldsymbol{p}_1 - \boldsymbol{p}_2)]} D_{(1)}^{\alpha\,\beta}(\boldsymbol{p}_0) \tag{29}$$

and

$$\boldsymbol{r}_0 := \boldsymbol{x}_i - \boldsymbol{x}_j\,, \qquad \boldsymbol{r}_1 := \boldsymbol{x}_i - \boldsymbol{y}_i\,, \qquad \boldsymbol{r}_2 := \boldsymbol{x}_j - \boldsymbol{y}_j \tag{30}$$

A (gradient) expansion in the momenta $\boldsymbol{p}_i, \boldsymbol{p}_j$ recasts the integral (28) into the form

$$R_2^{(1)}(\boldsymbol{x}_i, \boldsymbol{x}_j, t-s|\boldsymbol{y}_i, \boldsymbol{y}_j, 0) = \sum_{k=0}^{\infty} \mathcal{V}_{(tr.)}^{(k)}(\boldsymbol{x}_i, \boldsymbol{x}_j, t-s)\, e^{\frac{\varkappa\,(t-s)\,\Delta_2}{2}}(\boldsymbol{x}_i, \boldsymbol{x}_j|\boldsymbol{y}_i, \boldsymbol{y}_j) \tag{31}$$

The vertices $\mathcal{V}_{(tr.)}^{(k)}$'s are homogeneous *normal ordered* differential operators of degree k. Scaling analysis requires to focus on the terms $[\mathcal{V}_{(tr.)}^{(k)}]$ in the $\mathcal{V}_{(tr.)}^{(k)}$'s proportional to $\ln(\varkappa\,t)^{-1}$. Only the explicit expression of the first few of them is needed. The reason is the following. Upon inserting (31) into (26), the transition probability reduces to

$$e^{-t\,\mathcal{M}_n} = e^{\frac{\varkappa\,t}{2}\Delta_n} - \xi \left\{ \frac{\varkappa\,t\,\ln m}{2}\Delta_n - \sum_{i \neq j} \sum_{k} \frac{\mathcal{V}_{(tr.)}^{(k)}(\boldsymbol{x}_i, \boldsymbol{x}_j, t)}{2} \right\} e^{\frac{\varkappa\,t}{2}\Delta_n} + O(\xi^2) \tag{32}$$

Harmonic polynomials $\mathcal{H}_{\jmath\,\boldsymbol{l}}$ behave as eigenvectors of unit eigenvalue of the propagator:

$$e^{-t\,\mathcal{M}_n}\mathcal{H}_{\jmath\,\boldsymbol{l}} = \mathcal{H}_{\jmath\,\boldsymbol{l}} + \frac{\xi}{2} \sum_{i \neq j} \sum_{k} \mathcal{V}_{(tr.)}^{(k)}(\boldsymbol{x}_i, \boldsymbol{x}_j, t)\, \mathcal{H}_{\jmath\,\boldsymbol{l}} + O(\xi^2)$$

Suppose that $\mathcal{H}_{\jmath\boldsymbol{l}}$' is particle-permutation and translational invariant, as it is expected for inertial range tracer correlations. Suppose furthermore that $\mathcal{H}_{\jmath\boldsymbol{l}}$ contains powers of (components of) the variable $\boldsymbol{x}_i$ not lower than $k' \leq \jmath$ i.e that it has *minimal homogeneity degree* k'. Then such polynomial can only be reproduced by operators $[\mathcal{V}_{(tr.)}^{(k)}]$ with $k \leq k'$. The conclusion is that scaling analysis at order ξ requires in such a case the spectrum of the homogeneous differential operator

$$\mathcal{O}_{(tr.)}^{(k')} = \frac{1}{2} \sum_{k=0}^{k'} \sum_{i \neq j} [\mathcal{V}_{(tr.)}^{(k)}](\boldsymbol{x}_i, \boldsymbol{x}_j) \tag{33}$$

Physically, special interest have conserved quantities dominating the inertial range scaling of structure functions of the tracer. Symmetry and analyticity then restrict the corresponding scaling

analysis to the diagonalization of all the $\mathcal{O}^{(k)}_{(tr.)}$'s for $k \leq 2$. At this stage it is worth observing that the argument above also applies to the scaling analysis of the density field. The action of the transition probability on conservation laws of the dual space is

$$e^{-t\,\mathcal{M}_n^{\dagger}} \mathcal{H}_{\boldsymbol{J}} = \mathcal{H}_{\boldsymbol{J}} + \frac{\xi}{2} \sum_{i \neq j} \sum_{k} \mathcal{V}^{(k)}_{(de.)}(\boldsymbol{x}_i, \boldsymbol{x}_j, t)\, \mathcal{H}_{\boldsymbol{J}} + O(\xi^2) \tag{34}$$

where the operators $\mathcal{V}^{(k)}_{(de.)}$'s are obtained by replacing in (29) $R_2^{(1)}$ with

$$\tilde{R}_2^{(1)\dagger}(\boldsymbol{p}_0, \boldsymbol{p}_1, \boldsymbol{p}_2, t) := \frac{(p_{1\,\alpha} + p_{0\,\alpha})(p_{2\,\beta} - p_{0\,\beta})\, \left(1 - e^{-t\,\varkappa\,[p_0^2 + \boldsymbol{p}_0 \cdot (\boldsymbol{p}_1 - \boldsymbol{p}_2)]}\right)}{\varkappa\,[p_0^2 + \boldsymbol{p}_0 \cdot (\boldsymbol{p}_1 - \boldsymbol{p}_2)]} D^{\alpha\,\beta}_{(1)}(\boldsymbol{p}_0) \tag{35}$$

5.2. Scaling analysis for the tracer

The only vertex contributing to (33) is

$$\mathcal{V}^{(2)}_{(tr.)}(\boldsymbol{x}_i, \boldsymbol{x}_j, t) = \int \frac{d^d q}{(2\,\pi)^d}\, e^{\imath \boldsymbol{q} \cdot \boldsymbol{x}_{ij}}\, \frac{\left(1 - e^{-t\,\varkappa\,q^2}\right)\, D^{\alpha\,\beta}_{(1)}(\boldsymbol{q}; m)}{\varkappa\,q^2} \partial_{x_i^{\alpha}} \partial_{x_j^{\beta}} \tag{36}$$

By (24) in order to determine scaling it is sufficient to extricate the logarithmic asymptotics of (36) for

$$\frac{||\boldsymbol{x}_{ij}||^2}{\varkappa\,t} \ll 1 \tag{37}$$

The technical details of the calculation of the integral have no conceptual relevance and are outlined in Appendix A. After some algebra (Appendix C), it turns out that translational invariant harmonic polynomials of minimal homogeneity degree two can specify the limit of vanishing ξ of zero modes of (9a) if they coincide with the harmonic component of the homogeneous polynomials diagonalizing the operator

$$\begin{aligned} \mathcal{O}^{(2)}_{(tr.)} = \sum_{i \neq j} \frac{[\mathcal{V}^{(2:0)}](\boldsymbol{x}_i, \boldsymbol{x}_j)}{2} &= \frac{d + 1 - 2\,\wp}{2\,(d+2)\,(d-1)} \left\{ \mathcal{C}^{(2,n)}_{SO(d)} - \frac{d + (d-2)\,\wp}{d + 1 - 2\,\wp} \mathcal{C}^{(2,n)}_{SU(n-1)} \right\} \\ &- \frac{1}{2\,d} \left\{ \frac{[d + (d-2)\,\wp](d + 1 - n)[E_n + d\,(n-1)]}{(d+2)\,(d-1)\,(n-1)} + \wp(E_n - d) \right\} E_n \end{aligned} \tag{38}$$

where $\mathcal{C}^{(2,n)}_{SO(d)}$ and $\mathcal{C}^{(2,n)}_{SU(n-1)}$ are respectively the representations of the quadratic Casimir of $SO(d)$ and $SU(n-1)$ on the space of functions of n-particles in d-spatial dimensions (see Appendix B). As $\mathcal{C}^{(2,n)}_{SO(d)}, \mathcal{C}^{(2,n)}_{SU(n-1)}$ commute, the spectrum of $\mathcal{O}^{(2)}_{(tr.)}$ diagonalized on the space of translational and permutation invariant homogeneous polynomials of degree $\jmath$ can be classified in terms of the Gelfand-Zetlin patterns (see e.g. [15]). In essence, denoting by $\boldsymbol{a} = [a_1, \dots, a_{n-2}]$ $n-2$ non-negative integers satisfying

$$\sum_{i=1}^{n-2} a_i = \jmath \qquad \& \qquad a_i \geq a_{i'} \quad \text{for } i \geq i' \tag{39}$$

the eigenvalues of $\mathcal{C}^{(2,n)}_{SU(n-1)}$ read

$$\lambda_{SU(n-1)}(\boldsymbol{a})=\sum_{i=1}^{n-2}a_i\,(a_i-2\,i)+\frac{\jmath\,[n\,(n-1)-\jmath]}{n-1} \tag{40}$$

whilst the Casimir of $\mathcal{C}^{(2,n)}_{SO(d)}$ is

$$\lambda_{SO(d)}(\ell)=\ell\,(\ell+d-2) \tag{41}$$

for ℓ the total (hyper)-angular momentum. The conclusion is that permutation and translational invariant zero modes dominating the inertial range asymptotics of the n-point tracer correlation functions scale with exponents

$$\begin{aligned}\zeta^{(s)}_{n;1}([a_1,\dots,a_{n-2}],\ell)&=n\\&+\frac{\xi\,(d+1-2\,\wp)}{2\,(d+2)\,(d-1)}\left\{\ell\,(\ell+d-2)-\frac{d+(d-2)\,\wp}{d+1-2\,\wp}\left[\sum_{i=1}^{n-2}a_i(a_i+n-2\,i)-\frac{\jmath^2}{n-1}\right]\right\}\\&-\frac{\xi\,\jmath}{2\,d}\left\{\frac{[d+(d-2)\,\wp](d+1-n)[\jmath+d\,(n-1)]}{(d+2)\,(d-1)\,(n-1)}+\wp\,(\jmath-d)\right\}+O(\xi^2)\end{aligned} \tag{42}$$

Of particular interest is the irreducible $SO(d)$ isotropic zero mode governing the scaling of the structure function. This latter corresponds to the pattern of highest symmetry $[n,0,\dots,0]$ and scales with exponent

$$\zeta^{(s)}_{n;1}([n,0,\dots,0],0)=n-\xi\frac{n\,[d+n+2\,(n-2)\,\wp]}{2\,(d+2)}+O(\xi^2) \tag{43}$$

a result first obtained in [21] using operator product expansion (see also [14] for discussion).

5.3. *Scaling analysis for the density*

The adjoint action of the transition probability density brings about four interaction vertices. The first does not bring about any differential operation on harmonic polynomials:

$$\mathcal{V}^{(0)}_{(de.)}(\boldsymbol{x}_{ij},t):=-\int\frac{d^dq}{(2\,\pi)^d}\,e^{\imath\boldsymbol{q}\cdot\boldsymbol{x}_{ij}}\,\frac{q_\alpha q_\beta\,\left(1-e^{-t\,\varkappa\,q^2}\right)\,D^{\alpha\,\beta}(\boldsymbol{q};m)}{\varkappa\,q^2} \tag{44}$$

The other two, after use of the identities

$$D^{\alpha\,\beta}(\boldsymbol{q},m)q_\alpha q_\beta=-\wp\,D^{\alpha}_{\ \alpha}(\boldsymbol{q},m)\,q^2 \tag{45}$$

and

$$D^{\alpha\,\beta}(\boldsymbol{q},m)q_\alpha p_\beta=-\wp\,D^{\alpha}_{\ \alpha}(\boldsymbol{q},m)\,\boldsymbol{q}\cdot\boldsymbol{p} \tag{46}$$

reduce to

$$\mathcal{V}^{(1)}_{(de.)}(\boldsymbol{x}_i,\boldsymbol{x}_j,t)=-\,\imath\,t\,\wp\int\frac{d^dq}{(2\,\pi)^d}\,e^{\imath\boldsymbol{q}\cdot\boldsymbol{x}_{ij}}\,e^{-t\,\varkappa\,q^2}\,D^{\alpha}_{\ \alpha}(\boldsymbol{q};m)\boldsymbol{q}\cdot(\partial_{\boldsymbol{x}_i}-\partial_{\boldsymbol{x}_j}) \tag{47}$$

and

$$\mathcal{V}^{(2:1)}_{(de.)}(\boldsymbol{x}_i,\boldsymbol{x}_j,t)=\frac{\wp\,\varkappa\,t^2}{2}\int\frac{d^dq}{(2\,\pi)^d}e^{\imath\,\boldsymbol{q}\cdot\boldsymbol{x}_{ij}}e^{-\,t\,\kappa\,q^2}D^{\alpha}_{\alpha}(\boldsymbol{q},m)\left[\boldsymbol{q}\cdot(\partial_{\boldsymbol{x}_i}-\partial_{\boldsymbol{x}_j})\right]^2 \tag{48}$$

The fourth vertex is $\mathcal{V}^{(2:0)}_{(de.)}=\mathcal{V}^{(2:0)}_{(tr.)}$. As (47) and (48) are not logarithmic in $\varkappa\,t$

$$[\mathcal{V}^{(1)}_{(de.)}](\boldsymbol{x}_i,\boldsymbol{x}_j)=[\mathcal{V}^{(2:1)}_{(de.)}](\boldsymbol{x}_i,\boldsymbol{x}_j)=0 \tag{49}$$

the only new contribution relevant for scaling analysis comes from $\mathcal{V}^{(0)}_{(de.)}$ and is

$$[\tilde{\mathcal{V}}^{(0)}_{(de.)}](\boldsymbol{x}_{ij})=-\,\frac{\wp\,d}{2} \tag{50}$$

(36) indicates the existence of zero modes based at harmonic polynomials of zero degree in the limit $\xi\downarrow 0$. For these zero modes the operator

$$\mathcal{O}^{(0)}_{(de.)}=\frac{1}{2}\sum_{i\neq j}[\mathcal{V}]^{(0)}(\boldsymbol{x}_i,\boldsymbol{x}_j,t)=-\,\frac{\wp\,d\,n\,(n-1)}{2} \tag{51}$$

is diagonal and predicts the scaling dimension [21, 14]

$$\zeta_{n;2}=1-\,\xi\frac{\wp\,d\,n\,(n-1)}{2}+O(\xi^2) \tag{52}$$

The necessary condition that harmonic polynomials of minimal degree two need to satisfy in order to specify the limit of vanishing ξ of zero modes is to specify the harmonic component of the homogeneous polynomials diagonalizing the homogeneous operator

$$\begin{aligned}\mathcal{O}^{(2)}_{(de.)}=&-\frac{\wp\,d\,n\,(n-1)}{2}+\frac{d+1-2\,\wp}{2\,(d+2)\,(d-1)}\left\{\mathcal{C}^{(2,n)}_{SO(d)}-\frac{d+(d-2)\,\wp}{d+1-2\,\wp}\mathcal{C}^{(2,n)}_{SU(n-1)}\right\}\\&-\frac{1}{2\,d}\left\{\frac{[d+(d-2)\,\wp](d+1-n)[E_n+d\,(n-1)]}{(d+2)\,(d-1)\,(n-1)}+\wp(E_n-d)\right\}E_n\end{aligned} \tag{53}$$

Again, Gel'fand-Zetlin patterns satisfying (39) provide a natural way to classify the spectrum of $\mathcal{O}^{(2)}_{(de.)}$. The corresponding spectrum of scaling dimensions is

$$\begin{aligned}\zeta^{(s)}_{n;2}([a_1,\dots,a_{n-2}],\ell)=&\,n-\xi\frac{\wp\,d\,n\,(n-1)}{2}+\xi\frac{(d+1-2\,\wp)\,\ell\,(\ell+d-2)}{2\,(d+2)\,(d-1)}\\&-\xi\frac{d+(d-2)\,\wp}{2\,(d+2)\,(d-1)}\left[\sum_{i=1}^{n-2}a_i(a_i+n-2\,i)-\frac{\jmath^2}{\jmath-1}\right]\\&-\frac{\xi\,\jmath}{2\,d}\left\{\frac{[d+(d-2)\,\wp](d+1-n)}{(d+2)\,(d-1)\,(n-1)}[\jmath+d\,(n-1)]+\wp(\jmath-d)\right\}+O(\xi^2)\end{aligned} \tag{54}$$

6. Discussion

The general expressions (42), (54) are fairly complicated. It is convenient to discuss their significance in the particular non-trivial case $n=4$. This is the case of zero modes contributing

to fourth order structure and correlation functions. Restricting the focus on isotropic zero modes, the Gelf'and-Zetlin patterns diagonalizing $\mathcal{O}^{(2)}_{(tr.)}$, $\mathcal{O}^{(2)}_{(de.)}$ select two harmonic polynomials of degree four

$$\mathcal{H}_{([4,0],0)} \propto \sum_{\{i,i'\}} (\boldsymbol{x}_i - \boldsymbol{x}_{i'})^4 \\ -2 \sum_{\{\{i,j\},\{i,j'\}\}} (\boldsymbol{x}_i - \boldsymbol{x}_j)^2 (\boldsymbol{x}_i - \boldsymbol{x}_{j'})^2 + 6 \sum_{\{\{i,i'\},\{j,j'\}\}} (\boldsymbol{x}_i - \boldsymbol{x}_{i'})^2 (\boldsymbol{x}_j - \boldsymbol{x}_{j'})^2 \quad (55)$$

and

$$\mathcal{H}_{([2,2],0)} \propto -(2\,d+1) \sum_{\{i,i'\}} (\boldsymbol{x}_i - \boldsymbol{x}_{i'})^4 + (d+2) \sum_{\{\{i,j\},\{i,j'\}\}} (\boldsymbol{x}_i - \boldsymbol{x}_j)^2 (\boldsymbol{x}_i - \boldsymbol{x}_{j'})^2 \quad (56)$$

where the pairs $\{i,i'\}$ and $\{j,j'\}$ are assumed different, as well as the pairs $\{i,j'\}$ and $\{j,i'\}$. (55), (56) are the generalization to d-dimensions of the expressions given in [5] for $d=3$. They specify the harmonic component of the homogeneous polynomials diagonalizing $\mathcal{O}^{(2)}_{(tr.)}$, $\mathcal{O}^{(2)}_{(de.)}$. They differ from the full eigenstate by linear combinations of *slow modes* i.e. polynomials of the form $\phi_{\jmath',\boldsymbol{\imath},\jmath-\jmath'}$'s with $\jmath' < \jmath$ which are $\mathbb{L}^2$-orthogonal to the harmonic part on the hyper-sphere $\mathbb{S}^8$ ($\mathbb{S}^{d_n-1}$ in the general case). In this sense, harmonic polynomials are replicated by the $\mathcal{O}^{(k)}_{(tr.)}$,$\mathcal{O}^{(k)}_{(de.)}$.

A second relevant aspect [10, 22, 13] on which to pay attention is represented by the relations interweaving small and large scale scaling exponents. Consider the Green function associated to the translation invariant transition probability density P_n. The asymptotic expansion

$$M_n^{-1}(\boldsymbol{Y},\boldsymbol{Y}') = \sum_{i=0}^{\infty} \begin{cases} \phi_{i;1}(\boldsymbol{Y})\bar{\psi}_{i;1}(\boldsymbol{Y}') & Y < Y' \\ \bar{\psi}_{i;2}(\boldsymbol{Y}')\phi_{i;2}(\boldsymbol{Y}) & Y > Y' \end{cases} , \quad \bar{\psi}_{i;r}(\boldsymbol{Y}) := \int_0^{\infty} dt\, \psi_{i;r}(\boldsymbol{Y},t) \quad (57)$$

dictates

$$\mathsf{d}_{\bar{\psi}_{n;1}} + \mathsf{d}_{\phi_{n;1}} = \mathsf{d}_{\bar{\psi}_{n;2}} + \mathsf{d}_{\phi_{n;2}} = [2 - \xi - d(n-1)]\,\mathsf{d}_x \quad (58)$$

Bearing in mind the physical interpretation of the adjoint action of P_n these relations state that the knowledge of the scaling dimensions of inertial range zero modes of the tracer field entails that of large scale zero modes of the density field and vice-versa. For example, denoting with superscripts (s) and (l) respectively small and large scale exponents, the isotropic irreducible and reducible zero modes of the tracer four point correlation function give

$$\begin{aligned} &\zeta^{(s)}_{4;1}([4,0],0) = 4 - 2\,\xi\,\tfrac{d+4+4\,\wp}{d+2} + O(\xi^2) \\ &\zeta^{(s)}_{4;1}([2,2],0) = 4 - 2\,\xi\,\tfrac{d-2+\wp}{d-1} + O(\xi^2) \\ \Rightarrow \quad &\zeta^{(l)}_{4;2}([4,0],0) = -(3\,d+2) + \xi\,\tfrac{d+6+8\,\wp}{d+2} + O(\xi^2) \\ &\zeta^{(l)}_{4;2}([2,2],0) = -(3\,d+2) + \xi\,\tfrac{d-3+2\,\wp}{d-1} + O(\xi^2) \end{aligned} \quad (59)$$

whilst the corresponding object for the density yield

$$\begin{aligned} &\zeta^{(s)}_{4;2}([4,0],0) = 4 - 2\,\xi\,\tfrac{d+4+\wp\,[4+3\,d\,(d+2)]}{d+2} + O(\xi^2) \\ &\zeta^{(s)}_{4;2}([2,2],0) = 4 - 2\,\xi\,\tfrac{d-2+\wp\,[1+3\,d\,(d-1)]}{d-1} + O(\xi^2) \\ \Rightarrow \quad &\zeta^{(l)}_{4;1}([4,0],0) = -(3\,d+2) + \xi\,\tfrac{d+6+2\,\wp\,[4+3\,d(d+2)]}{d+2} + O(\xi^2) \\ &\zeta^{(l)}_{4;1}([2,2],0) = -(3\,d+2) + \xi\,\tfrac{d-3+2\,\wp\,[1+3\,d\,(d-1)]}{d-1} + O(\xi^2) \end{aligned} \quad (60)$$

As expected, in the incompressible limit of vanishing $\wp$ the relations become self-dual. Large scale zero modes may become manifest in the power law decay of certain statistical indicators at large point separations if the integral scale of the velocity field is much larger than the one of the forcing [22, 13]. A final observation is that a direct determination of scaling dimensions of large scale zero modes is naturally achieved by studying the time dependent martingales $\psi_{j\,\boldsymbol{l}\,0}$ rather their stationary counterparts $\bar{\psi}_{j\,\boldsymbol{l}\,0}$ which are not martingales. Within the approximation (37), straightforward algebra yields

$$\int d^{d_n} Y' \, P_n(\boldsymbol{Y}, \boldsymbol{Y}', t-s) \psi_{n\boldsymbol{l},0}(\boldsymbol{Y}', s) \stackrel{\frac{R^2}{\varkappa\tau} \ll 1}{=} \psi_{n\boldsymbol{l},0}(\boldsymbol{Y}, t) +$$
$$\xi \, \psi_{n\boldsymbol{l},0}(\boldsymbol{Y}, t) \left\{ -\frac{1}{2} \ln(\sqrt{\varkappa\tau}) \left(\frac{d_n}{2} + n \right) + \ln \frac{Y}{\sqrt{\varkappa\tau}} \frac{1}{\mathcal{H}_{\jmath\boldsymbol{l}}} \mathcal{O}^{(2)} \mathcal{H}_{\jmath\boldsymbol{l}} \right\} + \cdots + O(\xi^2) \tag{61}$$

where $\tau = t - s$ and dots stand for slow modes. Scaling prediction for time-dependent martingales can be then read from the prefactor of the logarithmic counter-terms needed to compensate the the $\ln(\varkappa\tau)^{-1}$ dependence in (61).

7. Conclusions

The technique illustrated is strongly reminiscent of the operator product expansion [23] previously also used to perform systematic calculations in the Kraichnan model (see e.g. [24, 21, 25, 12] and references therein). The differences are that the role of momentum cut-off's is played by the time and that instead of bases of abstract operator valued fields (composite operators) it makes use of a basis of harmonic polynomials or shapes $\mathbb{L}^2$ complete on the $\mathbb{S}^{d_n-1}$ hypersphere. These objects have a direct geometrical meaning and can be used in numerical simulations close to controlled or phenomenological Gaussian limits to test scaling properties as it was done in [13]. A final observation is that, at least in principle, the technique does not require linearity of the statistical field theory in order to be applicable. The reason is the following. In the thermodynamic formalism of field theory [23], connected n-point correlation functions $W^{(n)}$ are reconstructed by solving Hopf-like equations of the form

$$U^{(2)} \, W^{(n)} = \mathcal{F}_n(W^{(n-1)}, ..., W^{(1)}) \tag{62}$$

where $U^{(2)}$ stands for the set of *proper vertices* of order two acting on n-point connected correlation functions. The functionals $\mathcal{F}_n$'s in (62) are specified by functional derivatives of the Legendre transform connecting the free energy W (i.e. the generating function of connected correlations) to the thermodynamic potential (i.e. the generating function of proper vertices). Thus the $\mathcal{F}_n$'s depend upon all proper vertices of order less or equal to n. Once these latter ones are given, (62) defines a solvable hierarchy for the connected correlations. Thinking of the $U^{(n)}$'s as pseudo-differential operators, the generalization of the method presented in this paper consists in probing the scaling of "martingales" associated to the generalized propagators $U^{(2)-1}$ acting on n-particle functions. It must be, however, emphasized that deriving expressions of proper vertices is in general a very non-trivial task that can be most often carried out only when perturbative methods are applicable.

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Appendix A. Outline of the evaluation of the integrals

A convenient method to remove cut-offs from integrals associated to Feynman-diagram is to take the Mellin transform with respect to the cut-off (see e.g. [23]). For the velocity correlation this means

$$\tilde{D}_z^{\alpha\,\beta}(\boldsymbol{x}, m) = -\frac{D_0\,\xi\, m^{z-\xi}\,\hat{c}(z;\xi)}{z-\xi}\int\frac{d^d p}{(2\,\pi)^d}e^{\imath\,\boldsymbol{p}\cdot\boldsymbol{x}}\frac{\Pi(\hat{\boldsymbol{p}},\wp)}{p^{d+z}} \tag{A.1}$$

with $\Pi(\hat{\boldsymbol{p}},\wp) = (1-\wp)\delta^{\alpha\beta} - (1-d\,\wp)p^\alpha p^\beta/p^2$ and

$$\frac{\hat{c}(z;\xi)}{z-\xi} := \int_0^\infty \frac{dw}{w}\frac{\chi(w^2)}{w^{z-\xi}} \qquad \& \qquad \hat{c}(0;\xi) = 1 \tag{A.2}$$

All non-universal information of the cut-off function is stored in the residues of the function $\hat{c}$ for $\Re z \geq 2$. These residues are, however, of no relevance for universal properties of the system and further description of them is needed. The scope of this appendix is to apply the Mellin transform to the evaluation of the logarithmic asymptotics of the integrals introduced in the main text. The interested reader is referred to [12] for more details on this technique.

Appendix A.1. Evaluation of $\mathcal{V}^{(0)}_{(de.)}$

After taking the Mellin transform with respect to the two cut-off scales m and $\varkappa\,t$ (44) becomes

$$\tilde{\mathcal{V}}^{(0)}_{(de.);z,\zeta} = -\frac{D_o\,m^z\,\wp\,(d-1)\,\hat{c}(z;0)}{\varkappa\,z}\int\frac{d^d q}{(2\,\pi)^d}\,e^{\imath\boldsymbol{q}\cdot\boldsymbol{x}}\left\{\frac{1}{q^{d+z}} - \int\frac{d\zeta}{(2\,\pi\,\imath)}\frac{\Gamma(-\zeta)\,(\varkappa\tau)^\zeta}{q^{d+z-2\,\zeta}}\right\} \tag{A.3}$$

The momentum integral can be performed for

$$\Re(z-2\,\zeta) < 0 \quad \& \quad \Re\, z < 0 \tag{A.4}$$

The asymptotics for

$$\frac{x^2}{\varkappa\,\tau} \ll 1 \tag{A.5}$$

corresponds to a shift to the left of the contour of the inverse Mellin transform in ζ. In order to extricate the leading order of the asymptotics it is sufficient to approximate the inverse transform with the residue of the first pole which is encountered for $\Re\zeta = z/2$ ($\Re z < 0$):

$$\tilde{\mathcal{V}}^{(0:0)}_z = \frac{\wp\,d\,(m\,x)^z}{z}\left\{\frac{\Gamma\left(\frac{d}{2}\right)\Gamma\left(1-\frac{z}{2}\right)}{2^z\,z\,\Gamma\left(\frac{d+z}{2}\right)} - \frac{\Gamma\left(1-\frac{z}{2}\right)}{z}\left(\frac{\varkappa\,\tau}{x^2}\right)^{\frac{z}{2}}\right\} + \text{sub-leading} \tag{A.6}$$

Finally the limit of vanishing m requires shifting the contour of the inverse Mellin transform in z towards positive values of $\Re\, z$. The result is

$$\mathcal{V}^{(0:0)}(\boldsymbol{x}, t) = -\frac{\xi\,\wp\,d}{2}\ln\frac{x^2}{\varkappa\,\tau} + \text{sub-leading} \tag{A.7}$$

Appendix A.2. Evaluation of $\mathcal{V}^{(2)}_{(tr.)}$

Proceeding as above (36) becomes

$$\tilde{\mathcal{V}}^{(2:0)}_{(tr.);z\zeta}(\boldsymbol{x}_{12}) = -\frac{D_o\, m^z\, \hat{c}(z,0)}{z} \times$$
$$\int \frac{d^d q}{(2\,\pi)^d}\, e^{\imath \boldsymbol{q}\cdot\boldsymbol{x}_{12}} \left\{ 1 - \int \frac{d\zeta}{(2\,\pi\,\imath)} \frac{\Gamma(-\zeta)\,(\kappa\,\tau)^{\zeta}}{q^{-2\,\zeta}} \right\} \frac{\Pi^{\alpha\,\beta}(\hat{\boldsymbol{q}},\wp)}{q^{d+z+2}} \partial_{x_1^{\alpha}} \partial_{x_2^{\beta}} \tag{A.8}$$

The momentum integral exists for

$$\Re\zeta = z + 2 \;<\; 0 \qquad \& \qquad \Re z \;<\; -2 \tag{A.9}$$

The asymptotic expression of the logarithmic part of the vertex in the limit (A.5) is obtained by approximating the inverse Mellin transform in ζ with the residues of the poles for $\Re\zeta = (z+2)/2$ and $\Re\zeta = z/2$. The resulting pole in z is double with residue

$$\mathcal{V}^{(2)}_{(tr.)}(\boldsymbol{x}_i, \boldsymbol{x}_j, t) = -\varkappa\, t \ln \sqrt{m^2 \varkappa\, t}\, \partial_{\boldsymbol{x}_i} \cdot \partial_{\boldsymbol{x}_j}$$
$$+ \frac{2\,(1-\wp) + d - 1}{4\,(d+2)\,(d-1)} \mathcal{T}^{\alpha\,\beta}(\hat{\boldsymbol{x}}_{ij}; 2, \wp) \ln \frac{x_{ij}^2}{\varkappa\, t}\, x_{ij}^2 \partial_{x_i^{\alpha}} \partial_{x_j^{\beta}} + \text{sub-leading} \tag{A.10}$$

with

$$\mathcal{T}^{\alpha\,\beta}(\boldsymbol{x}; z, \wp) := \delta^{\alpha\,\beta} - \frac{z(1 - d\,\wp)}{d + z(1-\wp) - 1} \frac{x^{\alpha} x^{\beta}}{x^2} \tag{A.11}$$

Note that whenever acting on translational invariant functions

$$\sum_{i \neq j} \mathcal{V}^{(2)}_{(tr.)}(\boldsymbol{x}_i, \boldsymbol{x}_j, t) = \varkappa\, t \ln \sqrt{m^2 \varkappa\, t}\, \Delta_n$$
$$+ \frac{2\,(1-\wp) + d - 1}{4\,(d+2)\,(d-1)} \sum_{i \neq j} \mathcal{T}^{\alpha\,\beta}(\hat{\boldsymbol{x}}_{ij}; 2, \wp) \ln \frac{x_{ij}^2}{\varkappa\, t}\, x_{ij}^2 \partial_{x_i^{\alpha}} \partial_{x_j^{\beta}} + \text{sub-leading} \tag{A.12}$$

Appendix B. Definition of quadratic Casimir

Let i be the particle label $i = 1, \ldots, n$ and α, β be index of vector components of individual particles in $\mathbb{R}^d$. Upon defining

$$\mathfrak{g}_i^{\alpha\,\beta} := x_i^{\alpha} \partial_{x_i^{\beta}} \qquad \& \qquad \mathfrak{e}_i := \mathfrak{g}_{i\;\alpha}^{\alpha} \tag{B.1}$$

the dilation operator E_n acting on the n-particle space $\mathbb{R}^{n\,d}$ is

$$E_n = \sum_{i=1}^{n} \mathfrak{e}_i \tag{B.2}$$

On the same space, the representation of the quadratic Casimir of $SU(d)$ is

$$\mathcal{C}^{(2,n)}_{SU(d)} = \sum_{i\,j} \left(\mathfrak{g}_i^{\alpha\,\beta} - \frac{\delta^{\alpha\,\beta}}{d} \mathfrak{e}_i \right) \left(\mathfrak{g}_{j;\beta\,\alpha} - \frac{\delta_{\beta\,\alpha}}{d} \mathfrak{e}_j \right) = \sum_{i\,j} \operatorname{tr} \mathfrak{g}_i \mathfrak{g}_j - \frac{E_n^2}{d} \tag{B.3}$$

Using the dilation operator, this latter can be also written in terms of the quadratic Casimir of $SU(n-1)$:

$$\mathcal{C}^{(2,n)}_{SU(d)} = \mathcal{C}^{(2,n)}_{SU(n-1)} + \frac{d+1-n}{d\,(n-1)} E_n \left[E_n + d(n-1)\right] \tag{B.4}$$

The generators of $SO(d)$ for the i-th particle in d dimensions are

$$\mathfrak{l}_i^{\alpha\,\beta} := \mathfrak{g}_i^{\alpha\,\beta} - \mathfrak{g}_i^{\alpha\,\beta} \tag{B.5}$$

The generators of $SO(d)$ for n particles are additive functions of the generators of $SO(d)$ for single particle:

$$L_n^{\alpha\,\beta} = \sum_{i=1}^{N} \mathfrak{l}_i^{\alpha\,\beta} \tag{B.6}$$

The quadratic Casimir is

$$\mathcal{C}^{(2,n)}_{SO(d)} := \frac{1}{2} L^{\alpha\beta} L_{\alpha\beta} = \sum_{ij=1}^{n} \left\{ \mathrm{tr}\left(\mathfrak{g}_i \mathfrak{g}_j\right)\right\} - \mathrm{tr}\left(\mathfrak{g}_i \mathfrak{g}_j^t\right) \tag{B.7}$$

A excellent introduction to group theoretic methods for many-body systems can be found in [15].

Appendix C. Derivation of (38)

Omitting the Laplacian term as it vanishes on harmonic polynomials $[\mathcal{V}^{(2)}_{(tr.)}]$ reduces to

$$\sum_{i\neq j} [\mathcal{V}^{(2)}_{(tr.)}](\boldsymbol{x}_i, \boldsymbol{x}_j) = \\ \frac{d+2(1-\wp)-1}{2\,(d+2)\,(d-1)} \sum_{ij} \left[x_{ij}^2 \partial_{\boldsymbol{x}_i} \cdot \partial_{\boldsymbol{x}_j} - \frac{2\,(1-d\,\wp)}{d+2(1-\wp)-1} x_{ij}^{\alpha} x_{ij}^{\beta} \partial_{x_i^{\alpha}} \partial_{x_j^{\beta}} \right] \tag{C.1}$$

Unfolding the differences in $\boldsymbol{x}_{ij} = \boldsymbol{x}_i - \boldsymbol{x}_j$ and dropping differential terms vanishing on translation invariant functions, yields

$$\sum_{ij} \left[x_{ij}^2 \partial_{\boldsymbol{x}_i} \cdot \partial_{\boldsymbol{x}_j} - \frac{2\,(1-d\,\wp)}{d+2\,(1-\wp)-1} x_{ij}^{\alpha} x_{ij}^{\beta} \partial_{x_i^{\alpha}} \partial_{x_j^{\beta}} \right] \\ = -2 \sum_{ij} \left\{ \boldsymbol{x}_i \cdot \boldsymbol{x}_j \partial_{\boldsymbol{x}_i} \cdot \partial_{\boldsymbol{x}_j} - \frac{2\,(1-d\,\wp)}{d+2\,(1-\wp)-1} \frac{x_i^{\alpha} x_j^{\beta} + x_j^{\alpha} x_i^{\beta}}{2} \partial_{x_i^{\alpha}} \partial_{x_j^{\beta}} \right\} \tag{C.2}$$

Applying the definitions of appendix Appendix B, finally gives

$$\sum_{i\neq j} \boldsymbol{x}_{ij} \cdot \partial_{\boldsymbol{x}_{ij}} \mathcal{V}^{(2:0)}(\boldsymbol{x}_{ij}) = \frac{(d+1-2\,\wp)}{(d+2)\,(d-1)} \\ \left\{ \mathcal{C}^{(2,n)}_{SO(d)} - \frac{d+(d-2)\,\wp}{d+1-2\,\wp} \mathcal{C}^{(2,n)}_{SU(d)} - \frac{(d-1)(d+2)\wp}{d\,(d+1-2\,\wp)} E_n (E_n - d\,n) \right\} \tag{C.3}$$

The equality (B.4) permits to express this result in terms of $\mathcal{C}^{(2,n)}_{SU(n-1)}$.

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